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THE EFFECTS OF FREQUENCY ERRORS IN OFDM

J.H. Stott, M.A.

Research and Development Department
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Summary

An OFDM system can suffer from two types of frequency error: firstly, ***a frequency offset*** *(as might be caused by the tolerance of the local-oscillator (LO) frequency); secondly,* ***an error in the receiver master clock frequency*** *(which will cause the spacing of the demodulating carriers to differ from those transmitted,* ***and*** *the duration of the receiver integrate-and-dump process to differ from the reciprocal of the transmitted carrier spacing).*

The effects of both types of error are analysed. The analysis, applied to suitable synchronisation signals, could be used to help in the design of Automatic Frequency Control (AFC) strategies. It is also shown that, for symbols containing random data, either type of error causes inter-carrier interference (ICI). This effect is quantified as a function of the system parameters.

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1. INTRODUCTION 1

2. THE EFFECTS OF FREQUENCY OFFSET 1

    2.1 Definition of problem and terminology 1

    2.2. Trivial cases 3

        2.2.1 No frequency error at all 3

        2.2.2 Frequency offset by a multiple of the carrier spacing 3

    2.3 A useful integral 3

    2.4 A first step for the general case 3

    2.5 Quantification of noise-like ICI 4

        2.5.1 Analysis 4

        2.5.2 An approximation for small frequency errors 6

    2.6 Numerical assessment of parametric variation of ICI 6

        2.6.1 Variation of ICI with number of carriers 6

        2.6.2 Variation of ICI with fine frequency error 7

        2.6.3 Variation of ICI from carrier to carrier 7

    2.7 The phase term re-visited in greater detail 7

3. THE EFFECTS OF SAMPLING-FREQUENCY ERROR 10

    3.1 Definition of problem 10

    3.2 General analysis of reception 10

    3.3 Quantification of noise-like ICI 11

    3.4 Assessment of ICI 11

4. CONCLUSIONS 13

5. ACKNOWLEDGEMENT 13

6. REFERENCES 13

THE EFFECTS OF FREQUENCY ERRORS IN OFDM

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1. INTRODUCTION

The traditional explanation of how Orthogonal Frequency Division Multiplex (OFDM) systems work relies on the transmitted carriers and the demodulating carriers being exactly aligned with each other, *and* the receiver integrate-and-dump process taking place over a duration exactly equal to the reciprocal of the carrier spacing.[1] Under these two conditions *orthogonality* holds and there is no crosstalk between carriers.

If either of these conditions does not hold, then orthogonality is no longer perfectly maintained and a degree of inter-carrier crosstalk must be introduced. That this happens is well-known, but its precise quantification is not so readily accessible in the literature.

The OFDM system designer needs to know about the effects from two standpoints:

- to determine *how much* residual frequency error is permissible — to set tolerances.
- to understand exactly *how* errors affect the received signal — so as to design ways to deduce from the received signals (and appropriate reference signals, if needed) a suitable correction signal to drive an Automatic Frequency Control (AFC) loop.

Two kinds of frequency error have to be addressed, each normally requiring a separate AFC loop:

- *a frequency offset*, as might be caused by the tolerance on the local-oscillator (LO) frequency.
- *an error in the receiver master clock frequency*, which will cause the spacing of the de-modulating carriers to differ from those transmitted, *and* the duration of the receiver integrate-and-dump process to differ from the reciprocal of the transmitted carrier spacing.

This Report addresses these separately, in Sections 2 and 3 respectively.

2. THE EFFECTS OF FREQUENCY OFFSET

2.1 Definition of problem and terminology

Supposing that there are N points in the Discrete Fourier Transforms (DFTs) – usually implemented as Fast Fourier Transforms (FFTs) – used to generate and demodulate the Coded OFDM (COFDM) signal. Thus, there are potentially N carriers, although some at the extremes of the spectrum will be set to zero at the transmitter in order to provide a guard band for easier implementation of analogue filters.

For simplicity and clarity of presentation it will be assumed that the transmitted signal is continuous, without being divided into symbols. Provided that a guard interval of sufficient length is used, the operation of the receiver will be just the same as when (as in practice) the transmitted data changes from symbol to symbol. So let the transmitted signal be described as:

$$s(t)=\sum_{k=0}^{N-1} S_k e^{j(\omega_0+k\omega_S)t} \tag{1}$$

Where: $e^{j(\omega_0+k\omega_S)t}$ represents the kth carrier.
S_k is the complex amplitude of the kth carrier. Note that for some k (corresponding to the edges of the spectrum) $S_k = 0$;
ω_S is the carrier spacing (and $\omega_S = 2\pi/T_S$, where T_S is the active symbol period);
ω_0 is the frequency of the zeroth carrier (at the IF where it is processed).

Consider an *ideal* receiver which has no local oscillator (LO) noise. The ‘ideally-received’ signal *r(t)* is affected only by the channel (impulse response *h(t)*) — provided the guard interval is sufficiently long:

$$r(t)=\sum_{k=0}^{N-1} H_k S_k e^{j(\omega_0+k\omega_S)t} \tag{2}$$

where H_k is the complex frequency response of the channel at the frequency of the kth carrier. Strictly, the H_k *also* include a phase slope which is actually the consequence of any timing error rather than of the channel itself. Replacing H_k S_k by R_k, the received complex carrier amplitudes, the ideally-received signal becomes:

$$r(t)=\sum_{k=0}^{N-1} R_k e^{j(\omega_0+k\omega_S)t}. \tag{3}$$

At the receiver (using the DFT), *r(t)* is correlated with each of the possible carrier waveforms to determine the demodulated carrier amplitudes. Assume that the demodulating carriers are however subject to a frequency offset so that they take the form $e^{j(\omega_0+\Delta\omega_0+l\omega_S)t}$.

Let the frequency offset $\Delta\omega_0$ itself be thought of as having two parts:

$$\Delta\omega_0 = \delta\omega_0 + n\omega_S, \tag{4}$$

where n is an integer, chosen so that $n\omega_S$ represents a ‘coarse’ frequency error (of a whole number of carrier spacings), while $\delta\omega_0$ is the residual ‘fine’ error whose magnitude does not exceed one-half the carrier spacing (see Fig. 1).

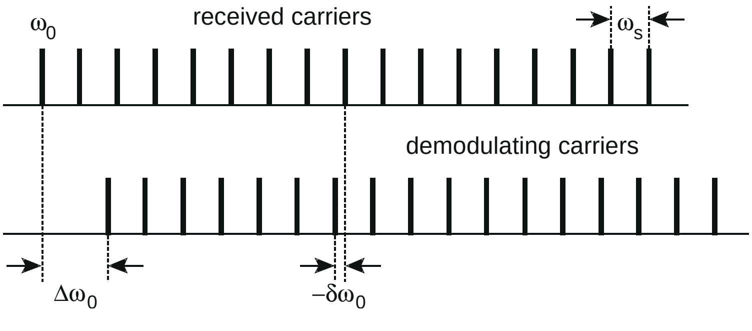

Fig. 1 - Illustrating the frequency offset $\Delta\omega_0$.

At the receiver, $r(t)$ is correlated with the *offset* carriers to obtain the following result for the complex amplitude which is output by the demodulator for the lth carrier:

$$\begin{aligned} Y_l &= \frac{1}{T_S}\int_0^{T_S} r(t)e^{-j(\omega_0+\Delta\omega_0+l\omega_S)t}dt \\ &= \frac{1}{T_S}\int_0^{T_S}\sum_{k=0}^{N-1} R_k e^{j\left(\overline{k-l-n}\,\omega_S-\delta\omega_0\right)t}dt. \end{aligned} \qquad (5)$$

2.2. Trivial cases

As a check that the method is correct, consider two special and very simple cases:

2.2.1 No frequency error at all

In this case, when $n = \delta\omega_0 = 0$, all the integrals in the summation except one vanish, which gives $Y_l = R_l$ as desired.

2.2.2 Frequency offset by a multiple of the carrier spacing

In this case, when $\delta\omega_0 = 0$, all the integrals in the summation, except one, vanish and $Y_l = R_{l+n}$ is obtained. Perfect reception is therefore realised, except that the data comes out of the wrong outputs of the FFT! This accords with what common sense would lead one to expect. If, though, the offset is large, the ‘edge’ carriers would fall outside the passband and some of the data would be lost. Some adjacent-channel interference would perhaps be imported on the other side of the spectrum. However, considering that a significant number of carriers are set to zero (to leave room for the analogue IF filter to roll off) it is possible, for an offset of a small integer number of carriers, that it would be perfectly valid to correct for possible data losses by re-routeing the data after the FFT.

2.3 A useful integral

At this point it is useful to consider the following integral, since it appears frequently in the analysis:

$$\begin{aligned} I &= \frac{1}{\tau}\int_T^{T+\tau} e^{jxt}dt \\ &= \frac{1}{\tau}\left[\frac{e^{jxt}}{jx}\right]_{t=T}^{t=T+\tau} = \frac{e^{jxT}\left(e^{jx\tau}-1\right)}{jx\tau} \\ &= \frac{e^{jx(T+\tau/2)}\left(e^{jx\tau/2}-e^{-jx\tau/2}\right)}{2(jx\tau/2)} \\ &= e^{jx(T+\tau/2)}\mathrm{sinc}(x\tau/2\pi). \end{aligned} \qquad (6)$$

(This is where the regularly-quoted sinc-characteristic frequency response originates.)

2.4 A first step for the general case

Applying the result of Section 2.3 to Equation 5, the result for the complex amplitude output by the lth-carrier demodulator is:

$$
\begin{aligned}
Y_l &= \sum_{k=0}^{N-1} R_k e^{j\left(\overline{k-l-n}\,\omega_S - \delta\omega_0\right)T_S/2} \operatorname{sinc}\left(\left(\overline{k-l-n}\,\omega_S - \delta\omega_0\right)T_S/2\pi\right) \\
&= e^{-j\delta\omega_0 T_S/2} \sum_{k=0}^{N-1} R_k e^{j(k-l-n)\pi} \operatorname{sinc}\left(k-l-n-\delta\omega_0/\omega_S\right) \\
&= e^{-j\delta\omega_0 T_S/2} \sum_{k=0}^{N-1} R_k (-1)^{(k-l-n)} \operatorname{sinc}\left(k-l-n-\delta\omega_0/\omega_S\right).
\end{aligned}
\tag{7}
$$

The $e^{-j\delta\omega_0 T_S/2}$ factor is a phase shift whose value corresponds to the *mean* phase of $-\delta\omega_0$ taken over the integration period. It applies equally to all l carriers — similar to the common phase error caused by phase noise. This holds the key to producing one means of obtaining a fine AFC and will be returned to in more detail later in this Report.

The summation has N terms in all, one of which corresponds to the demodulation of the 'expected' carrier. 'Expected' means: that carrier which is expected to come out of a particular output of the FFT, given that there is a 'coarse offset' of n carrier spacings.

Thus, the 'expected' carrier is demodulated as:

$$
Y_l = e^{-j\delta\omega_0 T_S/2} R_{l+n} \operatorname{sinc}\left(-\delta\omega_0/\omega_S\right) + (N-1)\text{ crosstalk terms.} \tag{8}
$$

It can be confirmed again that $Y_l = R_{l+n}$ for the case when $\delta\omega_0 = 0$. In this case, the crosstalk terms are zero — orthogonality is maintained.

If the 'fine' frequency error, $\delta\omega_0$, is non-zero, orthogonality is destroyed and there are $(N-1)$ crosstalk terms representing *intercarrier interference* (ICI).

It is clear that the behaviour of the ICI will depend on whether the information R_k on the carriers is *related* — as would be the case for a 'special' symbol, such as a phase-reference symbol — or *independent*, as would be the case for data symbols. To a degree, the transmitted information *is* actually related, through the deliberate introduction of redundancy by the error coding process. This is, however, diluted by the time the R_k values are considered for a practical, fading channel.

2.5 Quantification of noise-like ICI

2.5.1 Analysis

If the R_k are independent, then the $(N-1)$ crosstalk terms can be treated as added noise.

Let the statistics of the R_k be described as follows:

$$
\begin{aligned}
&\operatorname{var}\left(\operatorname{Re}\left(R_k\right)\right) = \sigma^2_{R_{\mathrm{Re}}}, \\
&\operatorname{var}\left(\operatorname{Im}\left(R_k\right)\right) = \sigma^2_{R_{\mathrm{Im}}}, \text{ and} \\
&\sigma^2_R = \sigma^2_{R_{\mathrm{Re}}} + \sigma^2_{R_{\mathrm{Im}}}
\end{aligned}
$$

Consider Equation 7, for the complex amplitude output by the lth-carrier demodulator:

$$Y_l = e^{-j\delta\omega_0 T_S/2}\sum_{k=0}^{N-1} R_k(-1)^{(k-l-n)}\operatorname{sinc}\left(k-l-n-\delta\omega_0/\omega_S\right).$$

Simplify this by writing $Y_l' = e^{j\delta\omega_0 T_S/2}Y_l$, to give

$$\begin{aligned} Y_l' &= \sum_{k=0}^{N-1} R_k(-1)^{(k-l-n)}\operatorname{sinc}\left(k-l-n-\delta\omega_0/\omega_S\right) \\ &= R_{l+n}\operatorname{sinc}(-\delta\omega_0/\omega_S) + Y_l'', \end{aligned} \tag{9}$$

where $Y_l'' = \sum_{\substack{k=0\\k\neq l+n}}^{N-1} R_k(-1)^{(k-l-n)}\operatorname{sinc}\left(k-l-n-\delta\omega_0/\omega_S\right)$,

and the qualification $k \neq l+n$ has been added to show that in the summation to form Y_l'', all values of k from 0 to $N-1$ are taken *except* $k = l+n$.

Y_l'' represents the noise-like ICI terms. Clearly:

$$\operatorname{var}\left(\operatorname{Re}\left(Y_l''\right)\right) = \sigma_{R_{\mathrm{Re}}}^2 \sum_{\substack{k=0\\k\neq l+n}}^{N-1}\operatorname{sinc}^2\left(k-l-n-\delta\omega_0/\omega_S\right),$$

$$\operatorname{var}\left(\operatorname{Im}\left(Y_l''\right)\right) = \sigma_{R_{\mathrm{Im}}}^2 \sum_{\substack{k=0\\k\neq l+n}}^{N-1}\operatorname{sinc}^2\left(k-l-n-\delta\omega_0/\omega_S\right),$$

and 'power' of

$$\begin{aligned} Y_l'' &= \operatorname{var}\left(\operatorname{Re}\left(Y_l''\right)\right) + \operatorname{var}\left(\operatorname{Im}\left(Y_l''\right)\right) \\ &= (\sigma_{R_{\mathrm{Re}}}^2 + \sigma_{R_{\mathrm{Im}}}^2)\sum_{\substack{k=0\\k\neq l+n}}^{N-1}\operatorname{sinc}^2\left(k-l-n-\delta\omega_0/\omega_S\right) \\ &= \sigma_R^2 \sum_{\substack{k=0\\k\neq l+n}}^{N-1}\operatorname{sinc}^2\left(k-l-n-\delta\omega_0/\omega_S\right). \end{aligned} \tag{10}$$

Note that, in this, it is assumed that all N carriers share a common received 'power' σ_R^2. In practice, some carriers at the edges are set to zero (so that the FFT can serve partly as a filter in both transmitter and receiver), so the number of terms taken into account in the summation must, in practice, be appropriately reduced.

Noting that the phase term makes no difference to the powers of 'wanted' carrier and ICI, the signal-to-noise ratio can be found (the σ_R^2 appears top and bottom, and cancels out):

$$\begin{aligned} \left.\frac{S}{N}\right|_{l\text{th demodulator}} &= \frac{\operatorname{sinc}^2\left(-\delta\omega_0/\omega_S\right)}{\sum_{\substack{k=0\\k\neq l+n}}^{N-1}\operatorname{sinc}^2\left(k-l-n-\delta\omega_0/\omega_S\right)} \\ &= \frac{\operatorname{sinc}^2(-\gamma)}{\sum_{\substack{k=0\\k\neq l+n}}^{N-1}\operatorname{sinc}^2(k-l-n-\gamma)}, \quad \text{writing } \gamma = \delta\omega_0/\omega_S. \end{aligned} \tag{11}$$

2.5.2 An approximation for small frequency errors

If γ is small (i.e. the ‘fine’ frequency error is a small part of the carrier spacing), some approximations can be made. Let m be a non-zero integer, then, with γ small:

$$
\begin{aligned}
\operatorname{sinc}(m-\gamma) &= \frac{\sin\left(\pi\,\overline{m-\gamma}\right)}{\left(\pi\,\overline{m-\gamma}\right)} \\
&\approx \frac{(-1)^m(-\pi\gamma)}{\pi m} \approx (-1)^{m+1}\frac{\gamma}{m} \\
\text{and} \qquad \operatorname{sinc}^2(m-\gamma) &\approx \frac{\gamma^2}{m^2}.
\end{aligned}
$$

For m zero and γ small, the approximation $\operatorname{sinc}^2\gamma \approx 1$ can be made.

An estimate can now be made of the signal-to-ICI-noise ratio for a carrier near the middle (assume that the ‘coarse’ frequency error is a number of carrier spacings, the total being small compared with the total number of carriers):

$$
\begin{aligned}
\left.\frac{S}{N}\right|_{\substack{l\text{th demodulator}\\ \text{(near middle)}}} &\approx \frac{1}{\displaystyle\sum_{\substack{k=0\\ k\neq l+n}}^{N-1} \frac{\gamma^2}{(k-l-n)^2}} \\
&\approx \frac{3}{\gamma^2\pi^2}, \quad \left(\text{using} \sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6} \text{ and assume large } N\right).
\end{aligned}
\tag{12}
$$

As an example, for $\gamma = 0.01$ the signal-to-ICI-noise ratio is estimated as 34.8 dB. Strictly, this is the result for very large N; for small N, the result should be more favourable (see next Section, 2.6).

2.6 Numerical assessment of parametric variation of ICI

Equation 11 can be evaluated numerically using *Mathematica*,*[2] although it can be rather slow to produce a result if the number of OFDM carriers is large. This application has been used to produce the examples that follow.

2.6.1 Variation of ICI with number of carriers**

Here, the approximate result of Equation 12 is verified. For the range of carrier numbers likely to be useful in practice, Fig. 2 shows that the change in signal-to-ICI-noise ratio is negligible. This result also serves to instruct that it is scarcely worth accounting for the precise number of ‘active’ carriers, as opposed to the total number.

---

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** In practice, some carriers at the edge of the spectrum are normally omitted in order to simplify filtering. The number of radiated carriers is therefore typically smaller than the number of points in the DFT. Although, in Section 2.1, *N* was defined as the DFT size, it can be seen that the form of the derivations is not altered if we re-interpret *N* as the number of carriers, forming a contiguous block in the frequency domain, which we actually used.

2.6.2 Variation of ICI with fine frequency error

Fig. 3 shows the variation with frequency error; in particular, it illustrates what happens in the range where the simple approximation given in Section 2.5.2 would not be valid. Note that values of γ whose magnitude exceeds 0.5 are of no interest, since a different FFT output should then be considered to represent the wanted carrier.

Fig. 2 shows that Fig. 3 can, for practical purposes, be used as a universal curve, independent of N. However, this does not mean that the signal-to-ICI-noise ratio is necessarily independent of N in a practical situation. If the signal bandwidth is kept constant while the number of carriers is changed, then γ ($= \delta\omega_0/\omega_s$) is proportional to N, if $\delta\omega_0$ is also kept constant. In such a circumstance, it is clear that the ICI becomes more severe as N increases.

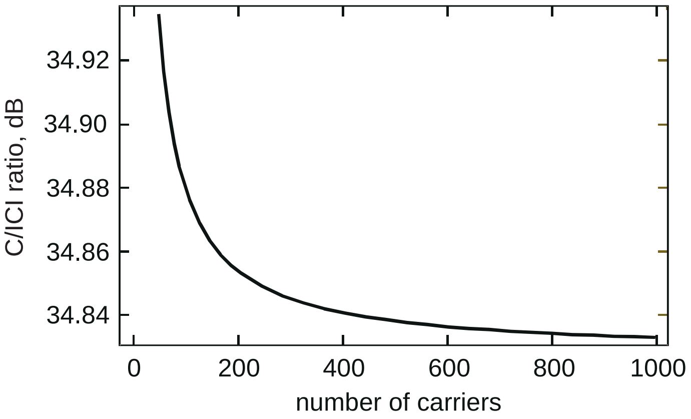


Fig. 2 - The variation of signal-to-ICI-noise ratio with the number of carriers; for a 'middle' carrier and fractional mistuning (γ = 0.01) *in an OFDM system.*

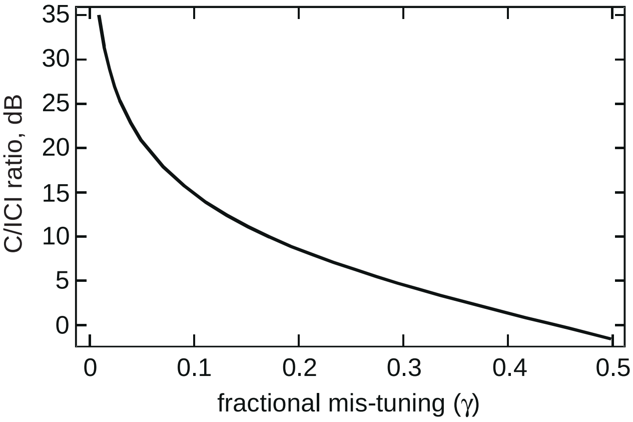


Fig 3 - The variation of signal-to-ICI-noise ratio with fractional mistuning; for a 'middle' carrier in a 128-carrier OFDM system.

2.6.3 Variation of ICI from carrier to carrier

The approximate result given in Section 2.5.2, Equation 12, was for a carrier close to the middle. Now a check is made as to how much the signal-to-ICI-noise ratio varies across the carriers. A simple example with just 128 carriers is considered in Fig. 4 *(overleaf)*. The fractional mistuning, γ, is 0.01 as before.

This figure shows that the very-edge carriers are nearly 3 dB better off, but most of the carriers are much the same as those in the middle. The slight asymmetry results from the inherently one-sided nature of a frequency offset. The calculation to make this presentation would take a wholly unbearable time if extended to a large number of carriers. Fig. 5 *(overleaf)* presents a compound picture for 8192 carriers and shows just the two edges and a small part of the middle. It should be noted that, even with this simplification, the calculation (using a fast Unix-based computer) took a considerable time to complete.

This expanded view of a situation with a large number of carriers confirms that the overwhelming majority of carriers essentially behave in a similar manner.

It is reassuring to note that these results appear to be in accordance with the analysis by Müller, given in Ref. 3.

2.7 The phase term re-visited in greater detail

Next, the behaviour of the phase factor which was identified in Section 2.4 is examined more closely.

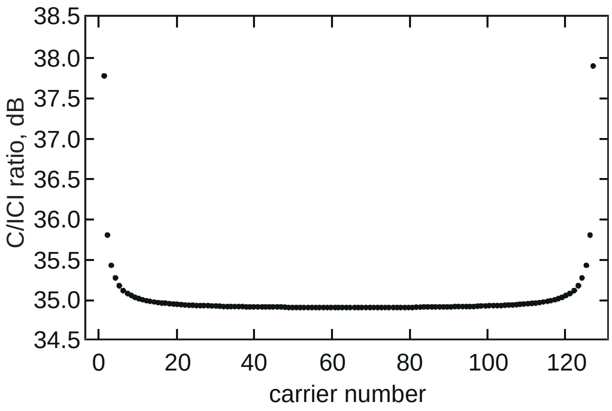


Fig. 4 - The variation of signal-to-ICI-noise ratio with carrier number; for fractional mistuning of 0.01 in a 128-carrier OFDM system.

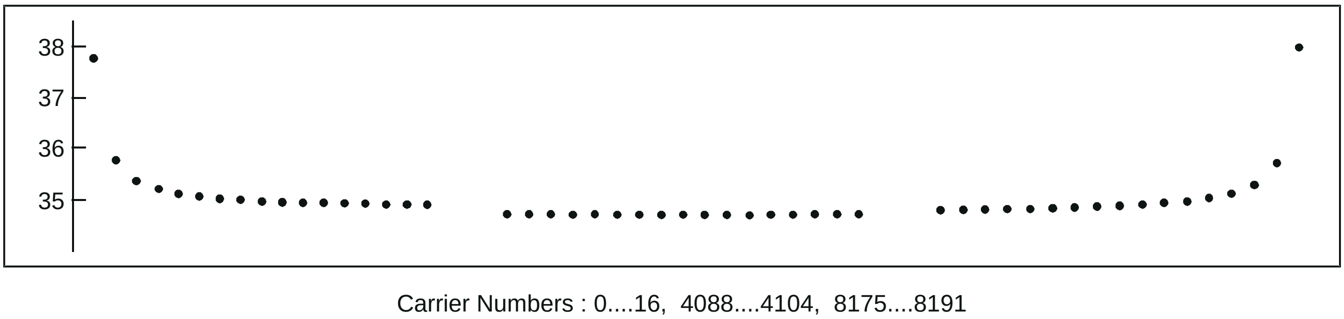


Fig. 5 - The variation of signal-to-noise ratio with carrier number; for fractional mistuning of 0.01 in an 8192-carrier OFDM system. Three groups of 16 carriers, from the two edges and the middle, are shown.

One might intuitively expect that the common phase error would advance or retard from symbol to symbol at a rate commensurate with the frequency error and the total symbol period (i.e. active period *and* guard interval). There are, however, complications to allow for: the frequency offset is continuous, but the carriers themselves are not (necessarily) continuous over the boundaries between symbols. (Note: the entire signal is, by design, *always* continuous across the boundary between the associated guard interval and active symbol period. The basis functions may also happen to be continuous between one symbol and the next for some subset only of the carriers — the subset being dependent on the exact proportion of the active symbol period which the guard interval represents. Whether the transmitted carriers are continuous over the symbol boundary also depends on the data conveyed by the symbols.)

An argument to account for the phases of the carriers now follows. Consider that one symbol (symbol zero, say) is transmitted from $-T_G$ to $+T_S$. The active symbol period will run from time 0 to $+T_S$, preceded by the guard interval; and the total symbol length is $T_T = T_G + T_S$. If the receiver is correctly synchronised in time, then the integration period will run from time 0 to $+T_S$. All the basis functions, $e^{jk\omega_S t}$, may be assumed to have the same value $1.0\angle 0°$ at both start and finish of the active symbol period. Indeed, this situation, for symbol zero, is what was analysed in the previous sections of this Report.

Later symbols will also be transmitted with the basis functions set to $1.0\angle 0°$ at both start and finish of the active symbol period. Similarly, the receiver's basis functions will be set to $1.0\angle 0°$ at the start and finish of the integration period. The RF/IF carrier on which they are modulated will however remain continuous, as will any frequency offset.

So the general formula for the *m*th symbol is:

$$s(t) = \sum_{k=0}^{N-1} S_{k,m} e^{j\left(\omega_0 t + k\omega_S \overline{t - mT_T}\right)}, \quad \left(mT_T - T_G\right) < t < \left(mT_T + T_S\right) \tag{13}$$

and, again (assuming that the receiver is correctly synchronised in time), if the guard interval is assumed to be long enough to encompass the most-delayed parts of the signal, the expression for the received signal can be simplified for the active symbol periods, within which the signal will be processed in the demodulators:

$$\begin{aligned} r(t) &= \sum_{k=0}^{N-1} H_k S_{k,m} e^{j\left(\omega_0 t + k\omega_S \overline{t - mT_T}\right)}, \quad mT_T < t < \left(mT_T + T_S\right) \\ &= \sum_{k=0}^{N-1} R_{k,m} e^{j\left(\omega_0 t + k\omega_S \overline{t - mT_T}\right)}. \end{aligned} \tag{14}$$

The time origin at the receiver has been redefined so as to nullify the time delay of the shortest path.

Now the general expression can be written for the *l*th demodulator output during the *m*th symbol:

$$\begin{aligned} Y_{l,m} &= \frac{1}{T_s} \int_{mT_T}^{mT_T + T_S} r(t) e^{-j\left(\overline{\omega_0 + \Delta\omega_0} t + l\omega_S \overline{t - mT_T}\right)} dt \\ &= \sum_{k=0}^{N-1} R_{k,m} \frac{1}{T_S} \int_{mT_T}^{mT_T + T_S} e^{j\left(\omega_0 t + k\omega_S \overline{t - mT_T}\right)} e^{-j\left(\overline{\omega_0 + \Delta\omega_0} t + l\omega_S \overline{t - mT_T}\right)} dt \\ &= \sum_{k=0}^{N-1} R_{k,m} e^{-j(k-l)\omega_S mT_T} \frac{1}{T_S} \int_{mT_T}^{mT_T + T_S} e^{j\left(-\delta\omega_0 + \overline{k-l-n}\omega_S\right)t} dt. \end{aligned} \tag{15}$$

Finally, Equation 6 is again applied:

$$\begin{aligned} Y_{l,m} &= \sum_{k=0}^{N-1} R_{k,m} e^{-j(k-l)\omega_S mT_T} e^{j\left(-\delta\omega_0 + \overline{k-l-n}\omega_S\right)mT_T} e^{j(k-l-n)\pi} e^{-j\delta\omega_0 T_S/2} \operatorname{sinc}\left(k - l - n - \delta\omega_0/\omega_S\right) \\ &= e^{-j\delta\omega_0 T_S/2} e^{-j\left(\delta\omega_0 + n\omega_S\right)mT_T} \sum_{k=0}^{N-1} R_{k,m} (-1)^{(k-l-n)} \operatorname{sinc}\left(k - l - n - \delta\omega_0/\omega_S\right). \end{aligned} \tag{16}$$

So exactly the same result as that obtained in Section 2.4, Equation 7, is repeated, except for the common phase term $e^{-j\delta\omega_0 T_S/2} e^{-j\left(\delta\omega_0 + n\omega_S\right)mT_T}$. The new part — $e^{-j\left(\delta\omega_0 + n\omega_S\right)mT_T}$ — represents the change in phase of the frequency offset, $\Delta\omega_0$, from symbol to symbol. This is clearly a reasonable result.

If the common phase for every symbol is measured, e.g. by using a subset of the carriers as pilot (reference) carriers, as has been proposed for combating LO phase noise, then its change could be measured over time to detect a frequency offset and derive an AFC signal. Alternatively, this measurement could be done once per frame, using a reference symbol; but the amount of frequency offset that can be detected without ambiguity is much less.

3. THE EFFECTS OF SAMPLING-FREQUENCY ERROR

3.1 Definition of problem

The problem to be analysed is that where the receiver's master clock frequency is not aligned with that of the transmitter. This causes two problems:

- the demodulating carriers have the wrong spacing;
- the period of the effective integrate-and-dump is also wrong.

For convenience, the terminology will be slightly redefined, so that the zeroth carrier is in tune at the demodulator, with progressively greater error for carriers further away. Also, let some kind of symmetry be maintained, so that this zeroth carrier is in the middle. If it is not, then, in effect, a frequency offset is being added as well and will thereby confuse the analysis and deductions.

So the transmitted signal becomes:

$$s(t)=\sum_{k=-N'}^{N'} S_k e^{j(\omega_0+k\omega_S)t},\ \text{ with a total of } N=2N'+1 \text{ carriers.}$$

As before, and with the same assumptions, the received signal is:

$$r(t)=\sum_{k=-N'}^{N'} R_k e^{j(\omega_0+k\omega_S)t},\ \text{ with } R_k=H_kS_k.$$

At the receiver, the demodulating carrier spacing and integration period become ω'_S and T'_S respectively, with $\omega'_S T'_S=2\pi$.

3.2 General analysis of reception

The output of the lth demodulator is given by:

$$\begin{aligned}
Y_l &= \frac{1}{T'_S}\int_0^{T'_S} r(t)e^{-j(\omega_0+l\omega'_S)t}dt \\
&= \frac{1}{T'_S}\sum_{k=-N'}^{N'} R_k \int_0^{T'_S} e^{j(k\omega_S-l\omega'_S)t}dt \\
&= \sum_{k=-N'}^{N'} R_k e^{j(k\omega_S-l\omega'_S)T'_S/2}\operatorname{sinc}\left((k\omega_S-l\omega'_S)T'_S/2\pi\right) \\
&= \sum_{k=-N'}^{N'} R_k e^{j(\zeta k-l)\pi}\operatorname{sinc}(\zeta k-l),\ \text{writing } \zeta=\frac{\omega_S}{\omega'_S}.
\end{aligned} \tag{17}$$

Provided that the sampling-frequency error remains very small, so that ζ is very close to unity, then this output can be considered to comprise a wanted term $R_l e^{j\pi l(\zeta-1)}\operatorname{sinc}\left(l(\zeta-1)\right)$ together with (N–1) unwanted ICI terms, in a similar way to the situation analysed in Section 2.

The wanted term can be seen to be reduced in amplitude by the sinc term, the reduction increasing with frequency error and as the carrier in question is displaced further from the centre. There is also a phase rotation which increases as the carrier in question is displaced further from the centre.

3.3 Quantification of noise-like ICI

When the information within a symbol is unrelated data, the unwanted ICI terms can be treated as noise. In a manner similar to that given in Section 2.5, an effective signal-to-ICI-'noise' ratio can be determined:

$$\left.\frac{S}{N}\right|_{l\text{th demodulator}} = \frac{\operatorname{sinc}^2\big(l(\zeta-1)\big)}{\sum\limits_{\substack{k=-N' \\ k\neq l}}^{N'} \operatorname{sinc}^2(\zeta k - l)}. \tag{18}$$

3.4 Assessment of ICI

The equation immediately above can be evaluated numerically for each carrier in turn for a particular value of sampling-frequency error. Unfortunately, this is quite a slow process using *Mathematica*. A simple example is shown in Fig. 6, in which the C/ICI ratio for each of 127 carriers is shown, for ζ of 1.0002, corresponding to a 200 ppm error in sampling frequency.

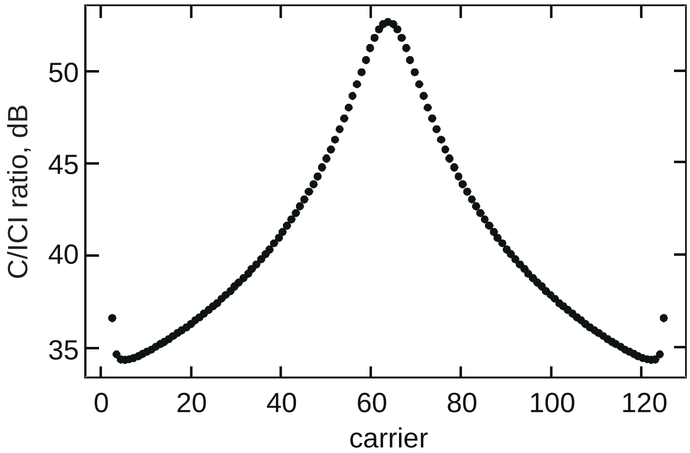
This curve has a rather different character from those presented in Section 2. Its properties can, however, be explained.

Recall that each demodulator has, in effect, a sinc^2 power response; thus, the contribution from other carriers depends on their degree of mistuning — if they were not mistuned, orthogonality would be satisfied and they would contribute no crosstalk. *Also*, because of the decaying nature of the sinc^2 characteristic, far-off carriers have less effect. (Consequently, in Section 2, with frequency offset, where all carriers are equally mistuned, the only significant variation is that edge carriers are better off because they have fewer nearby carriers.)

Applying this to the case of sampling-frequency error, note that the centre carrier is not mistuned at all, while the degree of mistuning increases as the edge is approached. Because the centre carrier is surrounded by carriers which are only slightly mistuned, it receives little ICI; but as the edge is approached, the general level of mistuning is greater, so the amount of ICI increases. The carrier at the very edge has the most-mistuned carriers near it — but only on one side of it (*being* at the very edge!), so it is less affected than its immediate neighbour. Thus, the worst-affected carrier is near, but not *at*, the

Fig. 6 - The variation of signal-to-ICI-noise ratio with carrier number, for ζ equal to 1.0002, corresponding to a 200 ppm error in sampling frequency in an OFDM system with 127 carriers.

Note that as a result of the means used to produce this graph, the carriers are numbered from 1 to 127, unlike *l* in the analysis.



edge. It is difficult to predict exactly which carrier is going to be the worst for any particular combination of ζ value and number of carriers used. This does rather preclude working out a set of worst-case parametric plots as was demonstrated in Section 2.

Fig. 7 shows another example, in this case with 6821 carriers and a value of 1.00001 for ζ (corresponding to 10 ppm). Only the first few carriers are shown in the figure. The middle (and best) carrier is not shown; it has a C/ICI ratio of 61.67 dB.

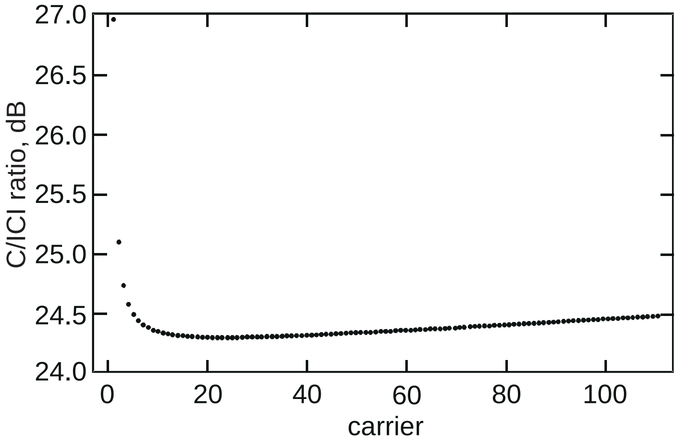


Fig. 7 - The variation of signal-to-ICI-noise with carrier number, for ζ equal to 1.00001, corresponding to a 10 ppm error in sampling frequency, with 6821 OFDM carriers in all.

Note that as a result of the means used to produce this graph, the carriers are numbered from 1 to 6821, unlike *l* in the analysis. Only the first 111 carriers are shown.

Both Figs. 6 & 7 suggest that the worst-off carrier suffers nearly 3 dB more ICI than the edge carrier. This is in accord with the qualitative reasoning above. It may therefore be possible to get a useful indication by making parametric plots for the *edge* carrier, and to interpret these under the assumption that the ‘worst’ carrier (whichever it may be) will have about 3 dB more ICI. Some values are plotted in Fig. 8 for a range of number-of-carriers and of sampling-frequency errors.

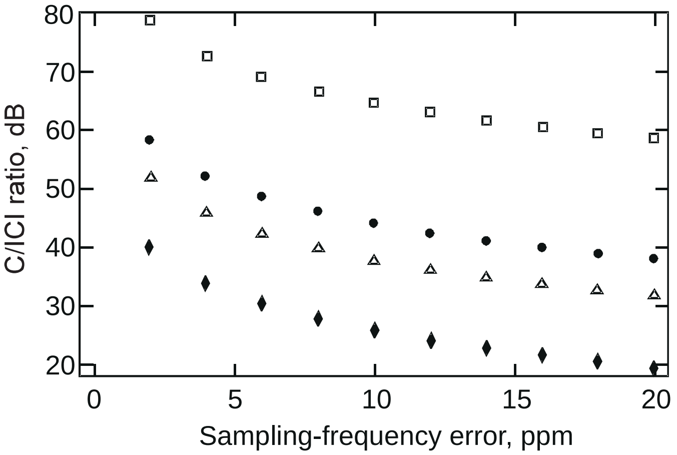


*Fig. 8 - The variation of signal-to-ICI-noise ratio with sampling-frequency error, for the **edgemost** carrier, when the total number of carriers takes the four different values.*

□ 101 carriers
● 1001 carriers
△ 2001 carriers
♦ 8001 carriers

Note that the worst-affected carrier may be expected to be 3 dB worse.

It may be deduced that little more than a 4 ppm sampling-frequency error can be tolerated by an 8000-carrier system if the C/ICI ratio of the worst carrier is required to exceed, say, 30 dB. However, this is what happens to the worst-afflicted carriers and should therefore be kept in perspective: it might, for example, be permissible to let the outer carriers more nearly approach the failure point and rely on the forward error-correction coding.

4. CONCLUSIONS

An OFDM can suffer from two types of frequency error:

- *a frequency offset*, as might be caused by the tolerance on the local-oscillator (LO) frequency.
- *an error in the receiver master clock frequency*, which will cause the spacing of the demodulating carriers to differ from those transmitted, *and* the duration of the receiver integrate-and-dump process to differ from the reciprocal of the transmitted carrier spacing.

Both of these situations have been analysed. The detailed equations may be used to study ways to derive synchronisation information from specially-inserted symbols — and to help in the design of such signals.

When the data being conveyed is random (as opposed to special channel-sounding or synchronisation symbols), then both types of frequency error may be shown to cause a noise-like inter-carrier interference (ICI). In this case, the degree of ICI can be characterised as follows:

- *a frequency offset* affects most carriers equally, with the very-edge carriers less affected.
- the effect of a *normalised* frequency offset varies little with the number of carriers, but is strongly dependent on the magnitude of the offset.
- the ICI resulting from a fixed *absolute* frequency offset increases with the number of carriers, if the system bandwidth is kept constant.
- *an error in the receiver master clock frequency* (in the absence of a frequency offset) affects the carriers unequally — the centre carrier suffers little while the worst-affected carrier lies close to, but not at, the edge.

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